§ 1.6 Algebraic subsets of the plane

classifying all algebraic subsets (reduce to find irr. ones)

Prop $F, G \in k[x,y]$ with $\gcd(F,G)=1$. Then $V(F,G) = V(F) \cap V(G)$ is finite

pf: $\gcd(F,G)=1$ in $k[x,y]$ $\Rightarrow$ $\gcd(F,G)=1$ in $k[x][y]$

$\Rightarrow$ $\gcd(F,G)=1$ in $k(x)[y]$ $(= \text{PID})$

$\Rightarrow \exists\ R, S \in k(x)[Y]$ s.t. $RF + SG = 1$

$\Rightarrow \exists\ D \in k[x] \setminus \{0\}$ s.t. $A = DR,\ B = DS \in k[x,y]$

$\Rightarrow$ $AF + BG = D$

$\forall\ (a,b) \in V(F,G) \Rightarrow D(a) = 0$ $\Rightarrow$ finite number of X-coordinates appeared among $V(F,G)$

similar for Y-coordinates $\Bigg\} \Rightarrow$ ✓.

Cor F = irr. in $k[x,Y]$. Suppose $V(F)$ is infinite. Then $I(V(F)) = (F)$ and $V(F)$ is irr.

($k = \mathbb{R}$ $F = x^2+y^2 \Rightarrow V(F) = \{(0,0)\}$.
$k = \bar{k} \Rightarrow V(F)$ is alway infinite.)

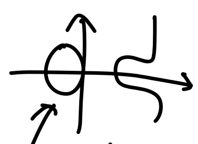
pf: $\forall\ G \in I(V(F)) \Rightarrow V(F,G) = V(F)$ is infinite

$\Rightarrow$ $\gcd(F,G) \neq 1$

$\Rightarrow$ $F \mid G$. $\Rightarrow$ $G \in (F)$.

Prop 1.5.1 $\Rightarrow$ $V(F)$ = irr.

Example



X not alg.

if $X = V(I) \subsetneq V(F) \Rightarrow (F) \subsetneq I$

$\forall\ G \in I \setminus (F) \Rightarrow V(I) \subseteq V(G,F)$ = finite

Cor $\#k=\infty$. irr. algebraic subsets of $\mathbb{A}^2(k)$ are

① $\mathbb{A}^2(k)$, ϕ

② points

③ irreducible plane curves $V(F)$, where F irr. and $V(F)$ is infinite.

Pf: $\forall$ V = irr. V = finite or $I(V)=0$ $\Rightarrow$ ✓

otherwise $\Rightarrow$ $\exists \underset{\neq 0}{F} \in I(V)$ = prime (WMA F = irr.)

$\Rightarrow$ $I(V)=(F)$ $\left(\text{or, } \forall G \in I(V) \setminus (F) \Rightarrow V \subset V(F,G) = \text{finite} \right)$ □

Cor $k=\bar{k}$, $F=F_1^{n_1}\cdots F_r^{n_r}$ (irr. decomposition). Then

1) $V(F)=V(F_1)\cup\cdots\cup V(F_r)$

2). $I(V(F))=(F_1\cdots F_r)$

Pf: 1) $V(F)=V(F_1^{n_1})\cup\cdots\cup V(F_r^{n_r})=V(F_1)\cup\cdots\cup V(F_r)$.

2) $I(\bigcup_i V(F_i))=\bigcap_i I(V(F_i)) \underset{\substack{1.6.2 \\ \#V(F_i)=\infty \\ \Uparrow \\ k=\bar{k}}}{=} \bigcap_i (F_i) \underset{F_i \neq F_j}{=} (F_1\cdots F_r)$

⑱

§1.7. Hilbert's Nullstellensatz.

describe V in terms of its definition polynormials

Assume $k=\bar{k}$ in this section!

Thm 1.7.1 (Weak Nullstellensatz) $k=\bar{k}$, $I \lhd k[x_1,\cdots,x_n]$ proper $\Rightarrow V(I)\neq\phi$.

反例 ($k\neq\bar{k}$)

Pf: WMA: $I=$ max. $\Rightarrow L=k[x_1\cdots x_n]/I$ field containing k.

$$\xRightarrow{\S 1a} \quad L=k \quad -(*)$$

$$\Rightarrow \forall i\ \exists a_i\in k \text{ s.t. } x_i-a_i\in I.$$

$$\Rightarrow I=(x_1-a_1,\cdots,x_n-a_n)$$

$$\Rightarrow V(I)=\{(a_1,\cdots,a_n)\}\neq\phi.$$

Thm 1.7.2 (Hilbert's Nullstellensatz) $k=\bar{k}$, $I\lhd k[x_1,\cdots,x_n]$. Then $I(V(I))=\sqrt{I}$

Thm $\Leftrightarrow$ if G vanishes wherever $F_1,\cdots,F_r$ vanish, then $\exists N>0$ s.t.

反例 ($k\neq\bar{k}$)

$$G^N\in(F_1,\cdots,F_r)$$

Pf (Rabinowitsch): $\sqrt{I}\subseteq I(V(I))$ clear.

$\forall G\in I(V(F_1,\cdots F_r))$.

$J:=(F_1,\cdots,F_r,X_{n+1}G-1)\subset k[x_1\cdots x_n,X_{n+1}]$

$$\Rightarrow V(J)\subseteq \mathbb{A}^{n+1}(k) \text{ empty}.$$

$$\xRightarrow[\text{nullstellensatz}]{\text{Weak}} \quad 1\in J$$

$$\Rightarrow \sum_i A_iF_i+B(X_{n+1}G-1)=1$$

$$\xRightarrow{Y=1/X_{n+1}} \quad Y^N=\sum_i C_i(X_1,\cdots,X_r,Y)F_i+D(X_1\cdots X_r,Y)(G-Y)$$

$$\Rightarrow \quad G^N=\sum_i C_i(X_1\cdots X_r,G)F_i\in(F_1,\cdots,F_r).\ \square$$

Cor 1.7.3 1) $\{\text{alg. sets}\} \xleftrightarrow{1:1} \{\text{radical ideals}\}$

反例 $(k \neq \bar{k})$ 2) $\{\text{irr. alg. sets}\} \xleftrightarrow{1:1} \{\text{prime ideals}\}$

$\{\text{points}\} \xleftrightarrow{1:1} \{\text{maximal ideals}\}$

3) $\{\text{irr. poly. } F\}/k^* \xleftrightarrow{1:1} \{\text{irr. hypersurface}\}$

4) $\underset{①}{V(I) = \text{finite}} \Leftrightarrow \dim_k k[x_1, \dots, x_n]/I < \infty$. ②

$\Rightarrow \# V(I) \leqslant \dim_k k[x_1 \cdots x_n]/I$. ③

Pf: ② $\Rightarrow$ ①, ③ :

$\forall P_1, \dots, P_r \in V(I)$

Problem 1.7 $\Rightarrow \exists F_i$ s.t. $F_i(P_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\forall \lambda_i \in k \quad \sum_i \lambda_i \overline{F_i} = 0$ in $k[x_1 \cdots x_n]/I$

$\Leftrightarrow \sum_i \lambda_i F_i \in I$

$\Rightarrow \sum_i \lambda_i F_i(P_j) = 0 \ \forall j$

$\Rightarrow \lambda_j = 0 \ \forall j$

$\Rightarrow \overline{F_1}, \dots, \overline{F_r}$ linear independent in $k[x_1 \cdots x_n]/I$

① $\Rightarrow$ ② : $V(I) = \{P_1, \dots, P_r\}$ $P_i = (a_{i1} \cdots a_{in})$

$F_j := \prod_{i=1}^{r} (X_j - a_{ij}) \in I(V(I)) \quad \forall j = 1, \dots, n$

$\overset{\text{Thm 1.7.2}}{\Rightarrow} \exists N$ s.t. $F_j^N \in I \quad \forall j = 1, \dots, n$

$\Rightarrow \dim_k k[x_1 \cdots x_n]/I \leqslant \dim_k \left(k[x_1 \cdots x_n] / (F_1^N, \dots, F_n^N) \right) = (rN)^n$